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Absolutely Pure Semimodules

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Abstract. In the present paper, a semimodule M over a semiring R is called absolutely pure if it is pure in every semimodule containing it as a subsemimodule. Some well-known properties of absolutely pure modules are extended to semimodules. We introduce and study two particular subclasses of absolutely pure semimodules, namely strongly absolutely pure (SAP) and finitely injective (f -injective) semimodules. When the semiring R is additively idempotent ,the SAP $R-$semimodules are exactly the f -injective semimodules.A characterization of Fieldhouse regular semimodules is obtained.

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Introduction

The notion of purity in module theory was defined in terms of tensor product.In [3 , Thm.2.4], P.M. Cohn proved that a submodule M of a left module N (over a ring R) is pure if every finite system of linear equations $H\overline{x} = \overline{m}$ with coefficients in R and parameters from M is solvable in M if it is solvable in N. Generally, if A and B are L-structures, where L is a first-order language, a homomorphism $f : A \to B$ is said to be pure if for any positive primitive formula ϕ and any tuple $\bar{a}$ from A, the validity of $\phi(f(\bar{a}))$ in B entails that of $\phi(\bar{a})$ in A [12] .This notion of purity was applied to semimodules over an arbitrary semiring and the existence of pure-injective semimodules was proved [14 ,Thm.3].In fact, one can easily show that a subsemimodule M of a left semimodule N (over a semiring R) is pure if every finite system of linear equations $H\overline{x} + \overline{m} = K\overline{x} + \overline{m}'$ with coefficents in R and parameters from M and with a solution in N already has a solution in M. In the present paper, a semimodule M is called absolutely pure if it is pure in every semimodule containing it as a subsemimodule.Some well-known properties of absolutely pure modules are extended to semimodules .For example , every semimodule has a maximal absolutely pure subsemimodule.We introduce and study two particular subclasses of absolutely pure semimodules, namely strongly absolutely pure (SAP) and finitely injective (f -injective) semimodules.A semimodule M is f- injective if and only if $M = \varinjlim X_i$,where the X_i are injective semimodules and the morphisms of the directed system $\{X_i\}$are injective.When the semiring R is additively idempotent ,the SAP $R-$semimodules are exactly the f -injective semimodules.A characterization of Fieldhouse regular semimodules is obtained.

1. Purity in Model theory

In this section, structure means structure for a given finitary similarity type and L is the first-order language of that similarity type. For the basic concepts of model theory we refer to [8]. Let us recall that if A and B are L-structures, a homomorphism $f : A \to B$ is said to be pure if for any positive primitive (p.p. for short) formula and any tuple $\bar{a}$ from A, the validity of $\phi(f(\bar{a}))$ in B entails that of $\phi(\bar{a})$ in A [12]. Note that every pure map is an isomorphic embedding, therefore these maps are also called pure embeddings. A substructure A of a structure B is called pure if the inclusion of A in B is pure.Elementary embeddings, that is, embeddings that preserve all first-order formulas, are clearly pure.

Lemma 1.1 [10]

Let A and B be two L-structures. The following conditions are equivalent for any embedding $f : A \to B$.

(i) f is pure

(ii) There is an elementary embedding $g : A \to C$ that factors through f (i.e. there is a homomorphism $h : B \to C$ such that $g = hf$).

Remark 1.1

In (ii) above g can be taken to be the diagonal embedding of A into an appropriate ultrapower of A [4, Th.6.4].

2. Purity in Semimodule Theory.

Let $R = (R; +, ., 0, 1)$ be a semiring, i.e. $(R; +, 0)$ is a commutative monoid with identity 0, $(R; ., 1)$ is a monoid with identity 1, for all $a, b, c \in R$, $a.(b+c) = a.b + a.c$ and $(b + c).a = b.a + c.a$, $0.r = 0 = r.0$ for all $r \in R$, and $0 \neq 1$. Let R be a semiring. A left R-semimodule is a commutative monoid $(M; +, 0)$ for which we have a function $R \times M \to M$, denoted by $(r, m) \longmapsto r\ m$ and called scaler multiplication, which satisfies the following conditions for all elements r and s of R and all elements m and n of M: (1) $(rs)m = r(sm)$; (2) $r(m + n) = rm + rn$; (3) $(r + s)m = rm + sm$; (4) $1\ m = m$; (5) $r\ 0 = 0 = 0\ m$. An element m of M is cancellable if $m + m' = m + m''$ implies that $m' = m''$. The semimodule M is cancellative if every element of M is cancellable. If every element $m \in M$ has an additive inverse $m' \in M$, the semimodule M is called an R-module.For the basic concepts of semirings and semimodules we refer to [7].Throughout this paper, semimodule means left semimodule over R. By ideal we mean a left ideal of R. By homomorphism, we mean an R-homomorphism.We consider the one-sorted first-order language L_R of left semimodules over a fixed arbitrary semiring R .Recall that a p.p. formula $\phi(\overline{x})$ is a formula of the form

$$\phi(\overline{x}) = \phi(x_1, ..., x_n) = \exists y_1...y_m(\bigwedge_{i=1}^{t} \Psi_i(\overline{x}, \overline{y})),$$

where $\overline{y} = (y_1, ..., y_m)$ and $\Psi_i(\overline{x}, \overline{y})$ are atomic formulas, $i = 1, ..., t$.

One can easily show that every atomic formula $\Psi(x_1, ..., x_n)$ of L_R is equivalent, modulo the theory of semimodules, to an equation

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n} b_i x_i,$$

where a_i, b_i are semiring elements. So, the p.p. formula $\phi(\overline{x})$ can be read as saying there are elements $\overline{y}$ such that $A\overline{x} + B\overline{y} = C\overline{x} + D\overline{y}$, where A, C are matrices (over R) of size $t \times n$, B, D are matrices of size $t \times m$, and $\overline{x}, \overline{y}$ are read as column matrices of semimodule elements. Let M, N be two R-semimodules and $f : M \to N$ be a pure embedding.This means that f is an injective R-homomorphism, and for any p.p. formula $\phi(\overline{x})$ and each tuple $\overline{m}$ from M, if there is a column matrix $\overline{b}$ (of elements of N) such that

$Af(\overline{m}) + B\overline{b} = Cf(\overline{m}) + D\overline{b}$
then there is a column matrix $\overline{c}$ (of elements of M) such that
$A\ \overline{m} + B\ \overline{c} = C\ \overline{m} + D\ \overline{c}$
where $f(\overline{m}) = (f(m_1), ..., f(m_k))$.

The following results follow from the definition of purity.

Lemma 2.1

Let $\phi(\overline{x})$ be a p.p. formula in L_R and M be an R-semimodule. Then
(i) $M \vDash \phi(\overline{0})$.
(ii) If $M \vDash \phi(\overline{a})$ and $M \vDash \phi(\overline{b})$, then $M \vDash \phi(\overline{a} + \overline{b})$.
(iiii) If $r \in C(R)$, the center of R, and $M \vDash \phi(\overline{a})$, then $M \vDash \phi(r\overline{a})$, where $r\overline{a} = r(a_1, ..., a_n) = (ra_1, ..., ra_n)$.
(iv) $\phi(M) = \{\overline{a} \in M^n : M \vDash \phi(\overline{a})\}$ is a submonoid of $(\mathrm{M}^n, +)$.
(v) If R is commutative, $\phi(M)$ is a subsemimodule of $(M^n, +)$.
(vi) If M is an R-module, $\phi(M)$ is a subgroup of $(M^n, +)$.

Lemma 2.2

Suppose that E,F and G are semimodules over a semiring R such that $E \subset F \subset G$.
(i) If E is pure in F and F is pure in G then$E$ is pure in G.
(ii) If E is pure in$G$ then$E$ is pure in F.

3. Absolutely Pure Semimodules

Let R be a semiring and M be an R-semimodule . If W is the subsemimodule of $M \times M$ defined by $W = \{(m, m) | m \in M\}$ then W induces an R-congruence relation $\equiv_W$ on $M \times M$, called the Bourne relation, defined by setting $(m, n) \equiv_W (m', n')$ if and only if there exist elements w and w' of W such that $(m, n) + w = (m', n') + w'$.

If $(m, n) \in M \times M$ then we write $(m, n)/W$ instead of $(m, n)/ \equiv_W$.The factor semimodule $M \times M/ \equiv_W$ is denoted by $M \times M/W$. Since for all $(m, n) \in M \times M$ we have $(m, n)/W + (n, m)/W = (0, 0)/W$, then $M \times M/W$ is an R-module.This left R-module, denoted by M^Δ, is called the R-module of differences of M.

Lemma 3.1 [7]

(i) A subsemimodule of a cancellative semimodule is cancellative .
(ii) Given a semimodule M, there is a homomorphism ξ_M of M into M^Δ, defined by $\xi_M(m) = (m, 0)/W$.
(iii) ξ_M is an embedding if and only if M is cancellative .

Definition

Let Γ be a class of R-semimodules. A semimodule $M \in \Gamma$ is said to be absolutely pure (AP),in Γ, if every embedding of M into a semimodule from Γ, is pure .When Γ is the class of all R-semimodules, M is said to be AP.

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Lemma 3.2

(i) A pure subsemimodule M of a module N is a module.

(ii) ξ_M is a pure embedding if and only if M is a module over the semiring R.

(iii) Any cancellative AP semimodule is a module .

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Proof.

(i) : Consider the p.p.formula $\phi(x_1) = \exists x_2(x_1 + x_2 = 0)$,and $m \in M$.Since N is a module ,then $N \vDash \phi(m)$,and so $M \vDash \phi(m)$.This means that m has an additive inverse in M.

(ii) : The "if" part follows from [7, Prop. 14.1] and Lemma 1. The "only if " part follows from (i).

(iii) : It follows from (i) and (ii) .□

Definition

Let M be semimodule over a semiring R. For two elements $a \in R$ and $m \in M$, the pair (a, m) is said to be compatible if the equation $ax = m$, has a solution in an extension of M.

Lemma 3.3

Let M be a cancellative semimodule over a cancellative semiring R. The following statements are equivalent for two elements $a \in R$ and $m \in M$:

(i) The pair (a, m) is compatible

(ii) There is a homomorphism $g : Ra \to M$ such that $g(a) = m$.

Proof.

(i) $\Rightarrow$ (ii) There is x_o in an extension of M such that $ax_o = m$. So, if $x = ra = ta \in Ra$, then $rm = rax_o = tax_o = tm$. Thus we may define $g : Ra \to M$ by $g(ra) = rm$. Of course, g is a homomorphism and $g(a) = g(1a) = m$.

(ii) $\Rightarrow$ (i): Let $I = Ra, g : I \to M$ and $g(a) = m$. We prove that there is an extension V of M and there is $x_o \in V$ such that $ax_o = m$ in V. Let i be the inclusion mapping of I into R and consider the homomorphism $\alpha = \xi_M g : I \to M \to M^{\triangle}$ and $\beta = \xi_R i : I \to R \to R^{\triangle}$. We define $f : I \to M^{\triangle} \times R^{\triangle}$ by $f(t) = (\alpha(t), -\beta(t))$ Note that $N = f(I)$ is a subsemimodule of $M^{\triangle} \times R^{\triangle}$ and N induces an R-congruence relation "Bourne relation" on $M^{\triangle} \times R^{\triangle}$. Let V be the factor semimodule $M^{\triangle} \times R^{\triangle}/N$. Let $\lambda : M \to M \times R \to M^{\triangle} \times R^{\triangle} \to V$; $u \mapsto (u, 0) \mapsto (\xi_M(u), 0) \mapsto (\xi_M(u), o)/N$ and $\mu : R \to M \times R \to M^{\triangle} \times R^{\triangle} \to V$;$r \mapsto (0, r) \mapsto (0, \xi_R(r)) \mapsto (0, \xi_R(r))/N$.

We show that λ is injective: Suppose $\lambda(u) = \lambda(v)$. Then $(\xi_M(u), 0)/N = (\xi_M(v), o)/N$ and so there are $n_1, n_2 \in N$ such that $(\xi_M(u), 0)+n_1 = (\xi_M(v), o)+ n_2$.If $n_1 = f(t_1), n_2 = f(t_2)$, then we get $\xi_M(u) + \alpha(t_1) = \xi_M(v) + \alpha(t_2)$ and $-\beta(t_1) = -\beta(t_2)$.

Since β is injective and M is cacellative, then $\xi_M(u) = \xi_M(v)$, and so $u = v$. This means that V is an extension of M. We prove that $\mu(1)$ is a solution of the equation $ax = m$ in V, i.e. $a\mu(1) = \lambda(m)$.

Since $(0, \xi_R(a)) + f(a) = (0, \xi_R(a)) + (\alpha(a), -\beta(a)) = (\alpha(a), 0) = (\xi_M(g(a)), 0) = (\xi_M(m), 0) = (\xi_M(m), 0) + f(0)$, then $(0, \xi_R(a)/N = (\xi_M(m), o)/N$.Thus, $\mu(a) = \lambda(m)$, and so $a\mu(1) = \lambda(m)$.□

Definition [1]

An R-semimodule M is called P-injective if for any principal ideal I of R and each homomorphism $g : I \to M$,there exists a homomorphism $f : R \to M$,which extends g.

Corollary 3.4

Every cancellative AP semimodule M over a cancellative semiring R is P-injective.

Proof.

Let $I = Ra$, $a \in R$,and $g : I \to M$ be a homomorphism . By the preceding Lemma the equation $ax = g(a)$ has a solution in an extension N of M , say , $\lambda : M \to N$ .Since M is AR , λ is pure and so the equation $ax = g(a)$ has a solution $m_o \in M$ .We define a homomorphism $h : R \to M$, by $h(r) = r\, m_o$.For any $x = ta \in I$, $g(x) = tg(a) = tam_o = h(x)$.Hence,$h$ extends g and so M is P-injective .□

Corollary 3.5

Let M be a cancellative semimodule over a cancellative semiring R .The following statements are equivalent:

(i) M is P-injective.

(ii)For any compatible pair $(a, m) \in R \times M$, the equation ⊛ "$ax = m$" has a solution in M .

Proof.

(i) $\Longrightarrow$ (ii) : Suppose (a, m) is compatible .By Lemma 3.3 , there is a homomorphism $g : Ra \to M$ such that $g(a) = m$. Since M is P-injective , there is a homomorphism $f : R \to M$, extends g. Observe that $ah(1) = h(a.1) = h(a) = g(a) = m$,and so $h(1) \in M$ is a solution of ⊛.

(ii) $\Longrightarrow$ (i) :Let $I = Ra$, $a \in R$, $g : I \to M$ be any homomorphism and $m_o = g(a)$.By Lemma 3.3, the equation $ax = m_o$ has a solution in an extension of M.Under the hypothesis (ii), this equation has a solution $u_o \in M$.We define a homomorphism $h : R \to M$,by $h(r) = ru_o$.One easily sees that h extends g.□

Proposition 3.6

Every pure subsemimodule$M$ of a P-injective semimodule N is P-injective

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Proof.

Let $I = Ra$, $a \in R$,and $g : I \to M$ be a homomorphism.Since $i : M \subset N$,and N is P-injective , there exists a homomorphism $f : R \to N$,which extends g .Hence , $i\ g(a) = g(a) = f(a) = m_o \in M$.Let $f\ (1) = n_o \in N$ and consider the equation $\circledast$: $ax = m_o$.Since $an_o = af(1) = f(a.1) = f(a) = m_o$, then $\circledast$ has a solution in N.Observe that $i : M \subset N$ is pure

and so $\circledast$ has a solution $u_o \in M$ (i.e. $au_o = m_o$).Now,define a homomorphism $h : R \to M$,by $h(r) = ru_o$.Since $h(ra) = rau_o = rm_o = rg(a) = g(ra)$,then h extends g and so M is P-injective.□

In [14 , Thm.5],it was proved that the first order theory T of cancellative semimodules over an arbitrary semiring R has the amalgamation property. As an application we have :

Proposition 3.7.

Every pure subsemimodule$M$ of a cancellative AP semimodule N is AP in the class of cancllative semimoules.

Proof.

Consider the following diagram , where f_1, f_2 are the identical inclusions ,f_1 is pure and H is a cancellative semimodule

$$\begin{array}{ccc} & & N \\ & \overset{f_1}{\nearrow} & \\ M & & \\ & \underset{f_2}{\searrow} & \\ & & H \end{array} \quad .$$

Let ($\circledast$) "$A\overline{x} + \overline{m} = B\overline{x} + \overline{m}'$ " be a finite system of linear equations with coefficents in R and parameters from M and with a solution in H .By[14 , Thm.5], there is a cancellative semimodule F and embeddings g_i , $i = 1, 2$,such that

the following diagram is commutative.

$$\begin{array}{ccccc} & & N & & \\ & \overset{f_1}{\nearrow} & & \overset{g_1}{\searrow} & \\ M & & & & F \\ & \underset{f_2}{\searrow} & & \underset{g_2}{\nearrow} & \\ & & H & & \end{array}$$

It follows that $(\circledast)$has a solution in F. Since $f_1$and $g_1$are pure , $(\circledast)$ has a solution in M and so M is AP in the class of cancellative semimodules.□

Theorem 3.8.

(i) If $X_0 \subset X_1 \subset \ ... \subset X_\beta \subset ..., \beta \prec \alpha$ is a chain of AP semimodules , where α is an ordinal , then the union of the chain is AP.

(ii) Every semimodule has a maximal AP subsemimodule .

Proof.

(i) Let $M = \cup X_\beta$ and suppose that $M \subset N$.Let $(\circledast)$: $A\overline{x} + \overline{m} = B\overline{x} + \overline{m}'$ be a finite system of linear equations with coefficents in R and parameters from M and with a solution in N.There is an ordinal γ such that the elements of the column matrices $\overline{m}$ and $\overline{m}'$ are in $X_\gamma \subset M \subset N$ .Therefore one can consider $\circledast$ as a finite system of linear equations with coefficents in R and parameters from X_γ and with a solution in N .Since $X_\gamma \subset N$, $\circledast$ has a solution in $X_\gamma \subset M$.

(ii) Given a semimodule E , consider the set Ω of all subsemimodules of E that are AP semimodules.Observe that Ω is not empty , for the zero semimodule belongs to Ω.Partially order Ω by inclusion.If $\mathcal{F}$ is a chain in Ω then $\cup\mathcal{F}$ is AP by (i).Now the result follows by applying Zorn's Lemma. □

4. SAP Semimodules

In [2] , Azumaya introduced the notion of locally split homomorphisms to study regular modules. Locally split submodules were introduced by Ramamurthi and Rangaswamy [9], by the name of strongly pure submodules, to study strongly absolutely pure (SAP) and finitely injective (f -injective in the sense of [13]) modules. In this section we extend these notions for semimodules over an arbitrary semiring.Let R be a semiring and M be an R-semimodule

M is said to be finitely injective (f -injective for short) if givn any injective homomorphism $F \rightarrow Y$,where F is a finitely generated semimodule, any homomorphism $F \rightarrow M$ can be extended to a homomorphism $Y \rightarrow M$.Note that every injective semimodule is f -injective. We call a subsemimodule M of a semimodule N strongly pure if to any finite set $\{m_1, ..., m_k\}$ of elements of M there exists a homomorphism $\alpha : N \rightarrow M$ such that $\alpha(m_i) = m_i$,$i = 1, ..., k$.Finally, a semimodule M is said to be strongly absolutely pure (SAP for short) if M is stongly pure in every R-semimodule containg it as a subsemimodule .

Proposition 4.1

Suppose that E, F and G are semimodules over a semiring R such that $E \subset F \subset G$.

(i) If E is strongly pure in F and F is strongly pure in G then E is strongly pure in G .

(ii) If E is strongly pure in G then E is strongly pure in F .

(iii) If E is strongly pure in F then E is pure in F.

(iv) If E is pure in F ,where F is a projective semimodule,then E is strongly pure in F.

(v) If E is pure in F ,where E is finitely generated and F is projective,then E is projective.

Proof.

(i) and (ii) are obvious. (iii) : Let ($\circledast$) "$A\overline{x}+\overline{u}=B\overline{x}+\overline{v}$ " be a finite system of linear equations with coefficents in R and parameters from E and with a solution $\overline{c}$ in F.Let $\{u_1,...,u_t,v_1,...,v_t\} \subset E$ be the elements of the column matrices $\overline{u}$, $\overline{v}$ .Under the hypothesis,there exists a homomorphism $\alpha : F\rightarrow E$, such that

$\alpha(u_i)=u_i$, $\alpha(v_i)=v_i$. It follows that$A\ \alpha(\overline{c})+\overline{u}=B\ \alpha(\overline{c})+\overline{v}$,and so $\alpha(\overline{c})$ is solution of $\circledast$ in E .(iv): If $f:E\subset F$ is pure, then by Lemma 1,there is an ultrafilter u over an infinite set I and a homomrphism $h:F\rightarrow E^I/u$ such that $h\ f=\delta$,where $\delta : E\rightarrow E^I/u$ is the diagonal embedding of E into an ultrapower of E.Let $\phi : E^I\rightarrow E^I/u$ be the canonical homomorpism.Since F is projective ,there exists a homomorphism $g:F\rightarrow E^I$ such that $\phi g=h$.Let $\{e_1,...,e_n\}$ be a finite

set of elements of E. For each e_k , $1\leq k\leq n$, $=\delta(e_k)=\ h(e_k)=\phi g\ (e_k)=g\ (e_k)/u$.Hence there exists a set $\Omega_k\in u$ suchs that $(g\ (e_k))(i)=e_k$ for all $i\in\Omega_k$.Let $\Omega=\cap\,\Omega_k\ \in u$, and define a homomorphism $\alpha=p_i\ g:F\rightarrow E^I\rightarrow E$, where p_i is the canonicl projection , $i\in\Omega$.It follows that $\alpha(e_k)=e_k$, $1\leq k\leq n$,and so E is strongly pure in F .

(v) : Let $\{e_1,...,e_n\}$ be a finite set of generators of E.Since $E\subset F$ is strongly pure , there exists a homomorphism $\alpha : F\rightarrow E$ such that $\alpha(e_i)=e_i$, $i=1,...,n$.It follows that E is a retract of F and so E is projective.□

Corollary 4.2

Every SAP semimodule is AP

Proposition 4.3

Let R be a semiring ,N be an R- module and M be a subsemimodule of N .The following statements are equivalent :

(i) $M\subset N$ is strongly pure.

(ii) $M\subset N$ is pure and for any element $x_o\in M$ there exists a homomorphism

$\alpha : N\rightarrow M$ such that $\alpha(x_o)=x_o$.

Proof.

(i)$\Longrightarrow$ (ii) follows from Proposition 4.1.

(ii)$\Longrightarrow$ (i) :By Lemma 3.2(i) M is a module . Let $\{x_1,...,x_n\}$ be any finite set of elements of M .We prove by induction, suppose $n\geq 2$ and our statement is true for $n-1$.This means that there is a homomorphism $\alpha : N\rightarrow M$ such that $\alpha(x_k)=x_k$ for $k=1,2,...,n-1$.Since $(x_n-\alpha(x_n)\ )\in M$, there is a $\beta : N\rightarrow M$ such that $\beta(x_n-\alpha(x_n))=x_n-\alpha(x_n)$ .Let $\delta=\alpha+\beta-\beta i\ \alpha$

$: N \rightarrow M$,where $i : M \subset N$ is the inclusion map.Then for any $k, k = 1, 2, ..., n-1, \delta(x_k) = \alpha(x_k) + \beta(x_k) - \beta i\ \alpha(x_k) = x_k + \beta(x_k) - \beta(x_k) = x_k$.And $\delta(x_k) = \alpha(x_n) + \beta(x_n) - \beta i\ \alpha(x_n) = \alpha(x_n) + \beta(x_n - \alpha(x_n)) = \alpha(x_n) + x_n - \alpha(x_n) = x_n$.Thus M is strongly pure in N .□

The following result connects finite injectivity with strong purity.

Proposition 4.4

Every f -injective semimodule is SAP.

Proof.

Let $M \subset N$, where M is an f -injective semimodule.For any finite set $T = \{m_1, ..., m_k\}$ of elements of M , let F be the semimodule generated by T.Consider the inclusion maps $f : F \subset N$ and $j : F \rightarrow M$.There exists a homomorphism $\alpha : N \rightarrow M$,such that $\alpha f = j$ .Observe that $\alpha(x) = x$,for all $x \in F$, and so M is SAP.□

Proposition 4.5

Every f -injective semimodule$M$ contains an injective hull of each of its finitely generated subsemimodule.

Proof.

Let $F \subset M$ be a finitely generated subsemimodule of M with an injective hull E. Consider the inclusion maps $i : F \subset M$ and $j : F \subset E$.Since M is f -injective semimodule,there exists a homomorphism $h : E \rightarrow M$ such that $hj = i$. Observe that h is injective since i is injective and j is essential.□

Corollary 4.6

Every finitely generated f -injective semimodule M is injective.

Theorem 4.7

(i) If $X_0 \subset X_1 \subset \ \ldots \subset X_\beta \subset \ldots, \beta \prec \alpha$ is a chain of f -injective semimodules , where α is an ordinal , then the union of the chain is f -injective.

(ii) Every semimodule has a maximal f -injective subsemimodule

(iii) A semimodule M is f -injective if and only if $M = \lim\limits_{\rightarrow} X_i$,where the X_i are injective semimodules and the morphisms of the directed system $\{X_i\}$are injective.

Proof.

We prove only (iii). Note that $M = \lim\limits_{\rightarrow}\{ F_i , \alpha_{ij} \}, i, j \in I$,where $\{ F_i \}$ is the family of all finitely generated subsemimodules of M and the morphisms $\{ \alpha_{ij} \}$are the inclusion maps.Since M is f -injective,it contains an injective hull $\widehat{F_i}$ of each F_i . One can easily check that $M = \cup\ \widehat{F_i}$ and the $\alpha_{ij} : F_i \rightarrow F_j$ induce injective homomorphisms $\widehat{\alpha_{ij}} : \widehat{F_i} \rightarrow \widehat{F_j}$,such that $\{\widehat{F_i}, \alpha_{ij}\}$ is a directed system and $M = \lim\limits_{\rightarrow}\{ \widehat{F_i} \}$.□

Remark. 4.1

If R is a ring ,then it is well-known that every R -module is contained in an injective R- module. However ,for arbitraty semirings R this not the case

; e.g. there are no nonzere injective $\mathbb{N}$ -semimodules. In [11],H.Wang proved that every R -semimodule has an injective hull ,in the case that R is additvely idempotent (i.e. a semiring satisfying $r + r = r$ for all $r \in R$).For these semirings we prove the convrse of Proposition 4.4.

Theorem 4.8

Let R be an additvely idempotent semiring.Then an R-semimodule M is f -injective if and only if M is SAP.

Proof.

Suppose M is SAP. Let $i : E \subset Y$,with E finitely generated by $\{e_1, ..., e_n\}$ and $g : E \rightarrow M$ .Let $\widehat{M}$ be the injetive hull of M and $j : M \rightarrow \widehat{M}$ .Since $\widehat{M}$ is injective , there is $h : Y \rightarrow \widehat{M}$, such that $hi = jg$ .Note that $\{g(e_k) : 1 \leq k \leq n\} \subset M$ and M is SAP . Hence there exists a homomorphism $\alpha : \widehat{M} \rightarrow M$ such that $\alpha(g(e_k)) = g(e_k)$, $1 \leq k \leq n$.One can easily show

that $\beta = \alpha h$ extends g and so M is f -injective.□

5. Regular Semimodules

A semiring R is said to be von Neumann regular if for each $a \in R$, there is some $b \in R$ such that $a = aba$. In [6], Fieldhouse generalized the concept of Von Neumann's regular rings to the module case : a module M (over a ring) is said to be regular if every submodule of M is pure in M . We extend this concept to semimodules ove an arbitrary semiring R.An R- semimodule M is said to be Fieldhouse regular if every subsemimodule of M is pure in M .

Theorem 5.1

For any R- semimodule M the following statements are equivalent :

(i) M is Fieldhouse regular.

(ii) Every finitely generated subsemimodule of M is pure in M.

If M is projective one can add :

(iii) Every finitely generated subsemimodule of M is a retract of M.

Proof.

(i)$\Longrightarrow$ (ii) is trivial.

(ii)$\Longrightarrow$ (i) : Let $E \subset M$. Note that $E = \varinjlim\{ F_i , \alpha_{ij} \}, i, j \in I$,where $\{ F_i \}$ is the family of all finitely generated subsemimodules of M and the morphisms $\{ \alpha_{ij} \}$are the inclusion maps.To show that E is pure in M , let (⊛) "$A\overline{x} + \overline{u} = B\overline{x} + \overline{v}$ " be a finite system of linear equations with coefficents in R and parameters from E and with a solution $\overline{m}$ in M. Let $\{u_1, ..., u_t, v_1, ..., v_t\} \subset E$ be the elements of the column matrices $\overline{u}$,$\overline{v}$.

There is $k \in I$ such that$\{u_1, ..., u_t, v_1, ..., v_t\} \subset F_k \subset E \subset M$ .Therefore one can consider ⊛ as a finite system of linear equations with coefficents in R and parameters fromF_k and with a solution in M.SinceF_k is pure M , ⊛ has a solution in

F_k . Thus E is pure in M and so M is Fieldhouse regular. Now suppose M is projective Fieldhouse regular and E is a finitely generated subsemimodule of M .Let $\{e_1, ..., e_n\}$ be a finite set of generators of E.By Proposition 4.1,

$E \subset M$ is strongly pure , thus there exists a homomorphism $\alpha : M \to E$ such that $\alpha(e_i) = e_i$, $i = 1, ..., n$.It follows that E is a retract of M .□

Corollary 5.2

For any semiring R consider the following statements :

(i) Every ideal of R is stongly pure in $_RR$

(ii) $_RR$ is Fieldhouse regular.

(iii) Every principal ideal of R is pure in R.

(iv) R is Von Neumann regular.

Then (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv).

Proof.

(i)$\Longrightarrow$(ii) and (ii) $\Longrightarrow$ (iii) are tivial . (ii) $\Longrightarrow$ (i) follows from Proposition 4.1.

(iii) $\Longrightarrow$ (iv) : For each $a \in R$, Ra is a pure subsemimodule of the R-semimodule $_RR$.The equation $ax + 0 = 0x + a$, with parameters from Ra, has a solution $(x = 1)$ in $_RR$. So, it has a solution in Ra. This means that there is $x_o = ra \in Ra$, for some $r \in R$, such that $ax_o = a$ .Thus $ara = a$, and so R is von Neumann regular

(iv) $\Rightarrow$ (iii): Suppose R is von Neumann regular and $I = Ra$. There is $b \in R$ such that $a = aba$. Let $e = ba$ and note that $e^2 = baba = ba = e$. Hence $I = \mathrm{Re}$.If $\theta : \mathrm{Re} \to {_RR}$ is the inclusion map and $\alpha :_R R \to \mathrm{Re}$, $\alpha(r) = re$, then $\alpha\theta = 1_{\mathrm{Re}}$

This means that $I = \mathrm{Re}$ is a retract of $_RR$, and, in particular, I is pure in R.□

Corollary 5.3

If every R-semimodule is AP then R is von Neumann regular.

Remark 5.1

For any ring R, the converse of the precding Corollary is true [5].On the other hand, the semiring $R = \mathbf{Q}^+$ is von Neumann regular and $M = {_RR}$ is not AP.

Remark 5.2

For any cancellative semiring R the following statements are equivalent :

(i) Every R- semimodule is AP.

(ii) Every cancellative $R-$semimodule is AP.

(iii) R is a regular ring.

Semimodules over rins are modules, so (iii) $\Longrightarrow$ (i) follows . (i) $\Longrightarrow$(ii) is obvious.

Now suppose (ii) , then $_RR$ is a module ,i.e. R is a ring.Indeed, R is a regular ring.

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